Surface wave propagation in a random layered medium

1987 J. Phys. A: Math. Gen. 20 2367

(http://iopscience.iop.org/0305-4470/20/9/024)

J. Phys. A: Math. Gen. 20 (1987) 2367-2377. Printed in the UK

Surface wave propagation in a random layered medium

B Lenoach

Department of Applied Mathematics, National Institute for Higher Education, Plassey Technological Park, Limerick, Ireland

Received 19 March 1985, in final from 21 November 1986

Abstract. We consider the propagation of surface Love waves in a random layered elastic medium. The material parameters in the model are stochastic functions of the vertical coordinates x_3: we include the general case when the wave propagates through a statistically inhomogeneous layer. We apply the averaging method for stochastic equations. The method essentially involves approximating the expectation of the solution with averages formed by integrating various moments of the random coefficients over 'long' times (long relative to the correlation time, see § 3). This method provides higher-order corrections to the purely asymptotic approximations which are often used to solve equations of this type. It is shown that the mean wave amplitude always attenuates exponentially on the depth scale $\varepsilon^2 x_3$ where ε is the fluctuation parameter. We analyse the dispersion relation for a statistically homogeneous layer and find a criterion which determines whether the mean phase velocity is decreased by the fluctuations.

1. Introduction

Random media are now widely studied in continuum mechanics (see, for example, Ishimaru 1982, Chow *et al* 1980); their value in the study of seismic wave propagation has been discussed by Hudson (1982). In this paper we consider the problem of Love waves in a layered medium consisting of a random elastic layer coupled to a homogeneous half-space. Specifically, the material parameters of the layer are taken to be random functions of the vertical coordinates x_3. We examine two distinct models as follows.

(i) The mean values of the material parameters are constant so that the layer is statistically homogeneous.

(ii) The physical situation (Aki and Richards 1980) suggets that it is interesting to also consider a model in which the mean values of the density and rigidity increase with the layer depth. In order to maintain a balance between a realistic model and a tractable one, the increase in the mean values is taken to be small, i.e. of the same order as the random perturbations themselves. This is done by assuming positive random coefficients. These processes are therefore non-stationary so the layer is now statistically inhomogeneous. In other words, the material parameters are no longer invariant under translation along the x_3 axis.

The first model may be regarded as a special case of the second. A quite different approach to the same general problem may be found in Hudson (1970).

We employ the averaging method for stochastic equations (Frigerio *et al* 1981). This method, introduced into the literature on random wave propagation in a previous paper (Lenoach 1983a) is suitable for stochastic problems in physics where the fluctuations, although small, are finite. Moreover, the method is designed to avoid the problem of secular terms which arises with regular perturbation theory and, on occasion, with the smoothing method (Hudson 1982).

The basic problem is formulated in § 2; in § 3 we give a brief account of our method. The asymptotic solution for (i) above, which we shall refer to as the statistically homogeneous model, is then given. We solve the more complicated second model to the same order in the fluctuation parameter ε: in both cases the random terms produce exponential attenuation on the depth scale $\varepsilon^2 x_3$. We also examine the dispersion relation obtained by requiring continuity of the motion-stress vector across the interface. This relation gives the shift in the phase velocity due to the fluctuations. It is shown that in general the phase shift and the amplitude attenuation are decoupled. Finally we derive, for the symmetric model, a criterion to determine whether the mean phase velocity is decreased by the fluctuations (as one might expect on physical grounds).

2. Formulation of the problem

We assume that the half-space $x_3 > H$ and the layer $0 \leqslant x_3 \leqslant H$ are isotropic linearly elastic solids and that no body forces are present. The equations of motion (Achenbach 1973) are then

$$\partial t_{ij}/\partial x_j = \rho\, \partial^2 u_i/\partial t^2 \tag{2.1}$$

where u_i, $i = 1, 2, 3$, are the displacement components, ρ denotes the material density and the summation convention is employed. Here that t_{ij} are the cartesian components of the Cauchy stress tensor which is related to the displacements via the isotropic stress-strain constitutive relations,

$$\begin{aligned} t_{ij} &= 2\mu e_{ij} + \lambda e_{kk}\delta_{ij} \\ 2e_{ij} &= \partial u_i/\partial x_j + \partial u_j/\partial x_i. \end{aligned} \tag{2.2}$$

Assume that

$$\begin{aligned} \rho(x_3) &= \rho_0(1 + \varepsilon\rho_1(x_3)) \\ \mu(x_3) &= \mu_0(1 + \varepsilon\mu_1(x_3)) \end{aligned} \tag{2.3}$$

where $\rho_1(x_3)$, $\mu_1(x_3)$ are random perturbations. Mathematically ρ_1 and μ_1 are stochastic processes on the probability space (Ω, P) indexed by $x_3 \in [0, H]$ (Wentzell 1981) with

$$\begin{aligned} &M_\rho(x_3) = \langle\rho_1\rangle M_\mu(x_3) = \langle\mu_1\rangle \\ &R_{\rho\rho}(x_3, x_3') = \langle\rho_1(x_3)\rho_1(x_3')\rangle \\ &R_{\mu\mu}(x_3, x_3') = \langle\mu_1(x_3)\mu_1(x_3')\rangle \end{aligned} \tag{2.4}$$

denoting the mean and correlation functions. ε is a small dimensionless positive constant characterising the size of the fluctuations: the angular brackets denote expectation (average), i.e. integration over Ω with respect to the measure P (Wentzell 1981). As we shall see, the detailed form of the averaging approximations becomes quite complicated, particularly in the general case of a statistically inhomogeneous layer. We shall therefore adopt the simplifying assumption that the coefficients μ_1 and ρ_1 are statistically independent. We are interested in the effect of random fluctuations on the usual Love-wave solution of equations (2.1) and (2.2) (Aki and Richards 1980).

Therefore we take the material parameters of the half-space to be the constants μ_2, ρ_2 and write

$$\begin{aligned} u_j &= \delta_{j2} \exp[ik(x_1 - ct)] V(x_3) \qquad && 0 \leq x_3 \leq H \\ &= \delta_{j2} \exp[ik(x_1 - ct)] W(x_3) && x_3 > H \end{aligned} \tag{2.5}$$

so that c is the phase velocity and k is the wavenumber. At this point, it is useful to clarify the role of the variables c and k. It is assumed that one of these quantities, k (say) is prescribed and the other is then determined by the dispersion relation which is itself derived from the boundary conditions, namely continuity of the displacement u_2 and shear stress t_{23} at the interface $x_3 = H$. It follows from the equations of motion that the shear stress t_{23} must have the form

$$t_{23} = T(x_3) \exp[ik(x_1 - ct)] \tag{2.6}$$

where $T(0) = 0$ because $x_3 = 0$ is a traction-free surface. In fact, as we shall see in § 6, the dispersion relation implies that c itself is a random variable. Let

$$\begin{aligned} \beta_0^2 &= \mu_0 \rho_0^{-1} \qquad && \beta_2^2 = \mu_2 \rho_2^{-1} \\ \sigma_0^2 &= (\langle c \rangle^2 / \beta_0^2 - 1) && \sigma_2^2 = (1 - \langle c \rangle^2 / \beta_2^2) \end{aligned} \tag{2.7}$$

where β_2 is the shear-wave velocity in the half-space. It can be shown that solutions with real frequency and wavenumber exist only if we assume that

$$\beta_0 < c < \beta_2 \tag{2.8}$$

which implies $\sigma_0^2 > 0$, $\sigma_2^2 > 0$. Define the complex motion-stress vector $r(x_3)$ as follows:

$$r(x_3) = \begin{pmatrix} V + iT/k\sigma_0\mu_0 \\ V - iT/k\sigma_0\mu_0 \end{pmatrix}. \tag{2.9}$$

Equations (2.1)-(2.5) give, to order ε^2,

$$dr/dx_3 = [L_0 + \varepsilon L_1(x_3) + \varepsilon^2 L_2(x_3)] r(x_3) \tag{2.10}$$

where

$$L_0 = \begin{pmatrix} -ik\sigma_0 & 0 \\ 0 & ik\sigma_0 \end{pmatrix} \qquad L_2 = \frac{-ik\sigma_0 \mu_1^2(x_3)}{2} \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \tag{2.11}$$

$$L_1 = \begin{pmatrix} \dfrac{ik\sigma_0}{2}\mu_1 + \dfrac{ik^3}{2k\sigma_0}\left(\mu_1 - \dfrac{\langle c \rangle^2}{\beta^2}\rho_1\right) & \dfrac{-ik\sigma_0}{2}\mu_1 + \dfrac{ik^2}{2k\sigma_0}\left(\mu_1 - \dfrac{\langle c \rangle^2}{\beta_0^2}\rho_1\right) \\ \dfrac{ik\sigma_0}{2}\mu_1 + \dfrac{ik^2}{2k\sigma_0}\left(\mu_1 - \dfrac{\langle c \rangle^2}{\beta^2}\rho_1\right) & \dfrac{-ik\sigma_0}{2}\mu_1 - \dfrac{ik^2}{2k\sigma_0}\left(\mu_1 - \dfrac{\langle c \rangle^2}{\beta_0^2}\rho_1\right) \end{pmatrix}.$$

In order to eliminate the first term on the right-hand side of equation (2.10) we define

$$q(x_3) = \exp(-L_0 x_3) r(x_3) \tag{2.12}$$

so that

$$\begin{aligned} dq/dx_3 &= [\varepsilon A_1(x_3) + \varepsilon^2 A_2(x_3)] q(x_3) \\ q(0) &= r(0) = V(0)\begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{aligned} \tag{2.13}$$

where $A_i(x_3) = \exp(-L_0 x_3) L_i(x_3) \exp(L_0 x_3)$, $i = 1, 2$. The point of these transformations is that the equations of motion are now in a form suitable for a direct application of the averaging method.

We emphasise that the parameters k, μ_2, ρ_2, μ_0 and ρ_0 are sure, i.e. non-random coefficients in the differential equation for the displacement and stress components $V(x_3)$ and $T(x_3)$. The constant rigidities μ_2, μ_0 and densities ρ_2, ρ_0 are known for a given model: k is also prescribed and c is then determined from the dispersion relation (see § 6). The random nature of the medium enters through the fluctuations μ_1 and ρ_1: it is the stochastic nature of μ_1 and ρ_1 which makes the solution q a vector of stochastic processes, i.e. each component is a stochastic process.

3. The extended averaging method

In this section we outline the results of the averaging method for stochastic equations (Frigerio *et al* 1981) and give the extension required to deal with the case when the mean values M_ρ and M_μ are functions of the depth variable x_3. In general one may consider an equation on IR^n of the form

$$\mathrm{d}f^\varepsilon(t)/\mathrm{d}t = \varepsilon A(t\omega)f^3(t) \tag{3.1}$$

with given initial data

$$f^\varepsilon(0) = f_0.$$

Here ω denotes a point in the sample space Ω (Wentzell 1981), the function $A(t)$ is assumed continuous and the vector f_0 is independent of ω, i.e. $\langle f_0\rangle = f_0$. $\varepsilon A(t)$ may be replaced by the power series $\Sigma_{k\geq 1}\, \varepsilon^k A_k(t)$ without additional difficulty.

The averaging method is concerned with asymptotic (in the sense described below) approximations to the average of the solution vector $\langle f^\varepsilon\rangle(t)$. Note that higher moments of the solution may be found, if required, by looking at appropriate outer products (Lancaster 1969) of the basic vector f.

Let us clarify the sense in which we use the term asymptotic here. In equation (3.1), the random term $A(t)$ is multiplied by the small parameter ε: in these circumstances random terms are significant only over long times. In particular, when we speak of the asymptotic approximation to the average $\langle f^\varepsilon\rangle(t)$ we have in mind the case of small fluctuations and large times, i.e. small ε and large values of t. It is in this case that stochastic effects are important: a lengthy treatment of this point may be found in the review of Papanicolaou (1978).

Before we give an outline of the relevant details of the averaging method, we pause to explain the physical meaning of the asymptotic approximation for the model described in § 2. Let ν be a typical correlation length in the layer, for example in the statistically homogeneous model one may take

$$\nu = \tfrac{1}{2}(\nu_\mu + \nu_\rho)$$

where

$$\nu_\mu = \int_0^\infty R_{\mu\mu}(r)\,\mathrm{d}r$$

and ν_ρ is defined similarly. Then asymptotic approximations to the equation of motion (2.13) are valid for small ε and large values of the dimensionless length $x_3' = x_3\nu^{-1}$.

There are essentially two separate cases to be considered with the averaging method: in the first one assumes that

$$\langle A(t_1)\ldots A(t_{2m+1})\rangle = 0 \qquad m = 0, 1, 2, \ldots \tag{3.2}$$

i.e. all odd moments of the coefficient matrix vanish. This will be true, for example, when the random coefficients ρ_1 and μ_1 are zero-mean Gaussian processes.

If equation (3.2) holds then the approximation, which becomes exact in the limit $\varepsilon \to 0$, $t \to \infty$ with $\varepsilon^2 t = \tau$ finite (Papanicolaou 1978), is

$$\langle f^\varepsilon \rangle(t) = \exp(\varepsilon^2 t G^{(2)}) f_0 + \mathrm{O}(\varepsilon^2) \tag{3.3}$$

where

$$G^{(2)} = \lim T^{-1} \int_0^T \mathrm{d}t \left(\langle A_2(t) \rangle + \int_0^t \mathrm{d}s \langle A_1(t) A_1(s) \rangle \right). \tag{3.4}$$

In the second case when $\langle A_1 \rangle(t)$ is a function of t, which means that equation (3.2) does not hold, the purely asymptotic approximation is

$$\begin{aligned} &\langle f^\varepsilon(t) \rangle \sim \exp(\varepsilon t G^{(1)}) f_0 + \mathrm{O}(\varepsilon) \\ &G^{(1)} = \lim_{T \to \infty} T^{-1} \int_0^T \langle A_1 \rangle(t)\, \mathrm{d}t. \end{aligned} \tag{3.5}$$

In this case it is evident that in a physical problem such as ours where ε is a small but finite parameter, the latter approximation is rather crude. The following more accurate approximation may be obtained from the averaging method:

$$\langle f^\varepsilon \rangle(t) \sim (1 + \varepsilon M^{(1)}(t)) \exp[\varepsilon t (G^{(1)} + \varepsilon G^{(2)})] f_0 + \mathrm{O}(\varepsilon^2) \tag{3.6}$$

where

$$M^{(1)}(t) = \int_0^t \mathrm{d}s [\langle A_1 \rangle(s) - G^{(1)}]$$

$$\begin{aligned} G^{(2)} = \lim_{T \to \infty} T^{-1} \int_0^T \mathrm{d}t \bigg(&\langle A_2(t) \rangle + \int_0^t \mathrm{d}s\, \mathrm{cov}(A_1(t), A_1(s)) \\ &+ \int_0^t \mathrm{d}s [\langle A_1 \rangle(t) - G^{(1)}][\langle A_1 \rangle(s) - G^{(1)}] + [G^{(1)}, \langle A_1 \rangle(s)]_- \bigg). \end{aligned}$$

We have used the notation

$$\mathrm{cov}(A_1(t), A_1(s)) = \langle A_1(t) A_1(s) \rangle - \langle A_1(t) \rangle \langle A_1(s) \rangle$$

and $[G^{(1)}, A_1(s)]_-$ denotes the commutator of $G^{(1)}$ and $A_1(s)$. Notice that, by definition, $G^{(1)}$ is just the leading term in $A_1(t)$. It follows that $M^{(1)}(t)$ will not contain any secular terms, i.e. terms which grow at a rate proportional to t. The formula for $G^{(2)}$ is also written in a way which shows that divergent leading terms can be expected to cancel each other out. We close this section with a remark on the accuracy of the asymptotic approximations outlined here. From a technical point of view it is very difficult to obtain concrete error estimates for asymptotic solutions to stochastic equations. Nevertheless, when we described our basic approximation schemes as being accurate to $\mathrm{O}(\varepsilon^2)$, we meant that the error in these approximations is known to be bounded above (Frigerio *et al* 1981) by $M\varepsilon^2$ for some finite positive number M whose precise value is an extremely complicated function of the many parameters in a model of this type. For example, M depends on the precise from of the correlation functions chosen and the values of the correlation lengths. The interested reader may find more technical details in the previously cited paper (Frigerio *et al* 1981). We now consider the statistically homogeneous model which corresponds to case (i) of § 1.

4. Solution of the statistically homogeneous model

In the statistically homogeneous model the random perturbations are centred, i.e.

$$M_\rho(x_3) = 0 = M_\mu(x_3) \tag{4.1}$$

and the correlation functions are stationary:

$$\begin{aligned}\langle\rho_1(x_3)\rho_1(x_3')\rangle &= R_{\rho\rho}(|x_3 - x_3'|)\\ \langle\mu_1(x_3)\mu_1(x_3')\rangle &= R_{\mu\mu}(|x_3 - x_3'|)\end{aligned} \tag{4.2}$$

in other words, the fluctuations are invariant under translations along the x_3 axis. The applicable approximation formulae are equations (3.3) and (3.4): if one defines the half Fourier transforms

$$\begin{aligned}S_{\rho\rho}(\alpha) &= \int_0^\infty \exp(\mathrm{i}\alpha r) R_{\rho\rho}(r)\,\mathrm{d}r\\ S_{\mu\mu}(\alpha) &= \int_0^\infty \exp(\mathrm{i}\alpha r) R_{\mu\mu}(r)\,\mathrm{d}r\end{aligned} \tag{4.3}$$

then a calculation gives the components of the mean motion-stress vector in the compact form

$$\begin{aligned}\langle V\rangle(x_3) &= V(0)\exp(-\beta\varepsilon^2 x_3)\cos(k\sigma_0 + \varepsilon^2\delta)x_3\\ \langle T\rangle(x_3) &= -k\sigma_0\mu_0 V(0)\exp(-\beta\varepsilon^2 x_3)\sin(k\sigma_0 + \varepsilon^2\delta)x_3\end{aligned} \tag{4.4}$$

where

$$\begin{aligned}\beta = \tfrac{1}{4}k\{2[S_{\mu\mu}(0) + \mathrm{Re}\ S_{\mu\mu}(2k\sigma_0)] + (\sigma_0^2 + \sigma_0^{-2})[S_{\mu\mu}(0) - \mathrm{Re}\ S_{\mu\mu}(2k\sigma_0)]\\ + \langle c\rangle^2/\beta_0^4[S_{\rho\rho}(0) - \mathrm{Re}\ S_{\rho\rho}(2k\sigma_0)]\}\end{aligned} \tag{4.5}$$

$$\delta = \frac{k\sigma_0}{2} - \frac{k^2\sigma_0^2}{4}\left[\left((1-\sigma_0^{-2})^2 + \frac{2\langle c\rangle^2}{\sigma_0^2\beta^2}(\sigma_0^{-2} - \sigma_0^{-4})\right)\mathrm{Im}\ S_{\mu\mu}(2k\sigma_0) + \left(\frac{\langle c\rangle^2}{\beta_0^2\sigma_0^2}\right)^2 \mathrm{Im}\ S_{\rho\rho}(2k\sigma_0)\right].$$

δ and β are the phase and amplitude modulations, respectively, caused by the fluctuations introduced in equation (2.3). Observe that the solution is characterised by the values of the transforms $S_{\rho\rho}(\alpha)$ and $S_{\mu\mu}(\alpha)$ at two wavenumbers, zero and $2k\sigma_0$. In addition, the correlation functions do not have to be specified beyond the requirement that they decrease rapidly enough so that the integrals in equation (4.3) exist. The solution given in equation (4.4) is valid for small ε and large values of the dimensionless length $x_3\nu^{-1}$ (see § 3). A more detailed physical interpretation of this solution will be given below.

5. The statistically inhomogeneous model

Experimental surface-wave data (Aki and Richards 1980) indicates that it may be more realistic to consider a model in which the mean values increase with the depth in the layer. One may retain the assumption of symmetric random perturbations so that ρ (say) is given by $\rho = \rho_0(1 + \varepsilon\rho_1)$ with $\langle\rho_1\rangle = 0$ and $\rho_0(x_3)$ is a sure function which is compatible with experimental estimates of the density profile.

An alternative approach, adopted here, is to keep ρ_0 constant and incorporate the heterogeneous nature of the medium in the random terms. Our approach has the advantage that the sure problem ($\varepsilon = 0$) is simple and therefore has an explicit solution. On the other hand, it is restricted in the sense that the mean $\langle\rho\rangle = \varepsilon\langle\rho_1\rangle(x_3)$ is proportional to ε. In other words, the increase in the mean is small, of the same order as the fluctuations themselves whereas one may put $\rho_0 = \rho_0(x_3)$ without this constraint. In the latter case, however, it may be necessary to introduce a small parameter in order to solve the heterogeneous sure problem. Indeed, a standard method for solving surface-wave problems in a vertically inhomogeneous medium is numerical integration of the equations of motion and it is difficult to implement this approach when fluctuations are present. Therefore our approach should be regarded as a model of a complicated physical situation: we do not claim that it is equivalent to the alternative approach described above.

A relatively simple method of biasing the random material parameters μ_1 and ρ_1 in order to model the statistically inhomogeneous layer is to require that the processes μ_1 and ρ_1 be non-negative. It turns out that the simplest choice is when ρ_1 and μ_1 are squares, i.e.

$$\rho_1(x_3) = X^2(x_3) \qquad \mu_1(x_3) = Y^2(x_3) \tag{5.1}$$

where $X(x_3)$ and $Y(x_3)$ are independent centred Gaussian processes

$$\begin{aligned} &\langle X\rangle = 0 = \langle Y\rangle \\ &\langle X(x_3)X(x_3')\rangle = R_{XX}(x_3, x_3') \\ &\langle Y(x_3)Y(x_3')\rangle = R_{YY}(x_3, x_3'). \end{aligned} \tag{5.2}$$

We do not yet have to specify the correlation functions beyond the requirement that, for fixed x_3 (say), $R_{XX}(x_3, x_3')$ and $R_{YY}(x_3, x_3')$ are rapidly decreasing functions of x_3'. The mean values are now

$$M_\rho(x_3) = R_{XX}(x_3, x_3) \qquad M_\mu(x_3) = R_{YY}(x_3, x_3') \tag{5.3}$$

and the Gaussian expansion of higher moments in terms of the first two moments (Wentzell 1981) gives

$$\begin{aligned} \operatorname{cov}(\rho_1(x_3), \rho_1(x_3')) &= \langle\rho_1(x_3)\rho_1(x_3')\rangle - \langle\rho_1(x_3)\rangle\langle\rho_1(x_3')\rangle \\ &= 2R^2_{XX}(x_3, x_3') \\ \operatorname{cov}(\mu_1(x_3), \mu_1(x_3')) &= 2R^2_{YY}(x_3, x_3') \\ \operatorname{cov}(\mu_1(x_3), \rho_2(x_3')) &= 0. \end{aligned} \tag{5.4}$$

In the author's thesis (Lenoach 1983b) it is shown that, in accordance with what one might expect on physical grounds, no qualitative feature of the solution depends on whether squares or absolute values are chosen to represent ρ_1 and μ_1 although the latter are considerably more awkward to work with. Therefore we henceforth work with the choice expressed in equation (5.1). Combining equations (3.5) and (5.1) gives

$$G^{(1)} = 0. \tag{5.5}$$

This result simplifies the calculations considerably because equation (5.5) implies that the relevant approximation formula is

$$\langle f^\varepsilon(t)\rangle \sim (1 + \varepsilon M^{(1)}(t)) \exp(\varepsilon^2 t G^{(2)}) f_0. \tag{5.6}$$

The small x_3 correction term has the simple form

$$M^{(1)}(x_3)=\begin{pmatrix} 0 & M(x_3) \\ M^*(x_3) & 0 \end{pmatrix} \tag{5.7}$$

where * denotes the complex conjugate and

$$M(x_3)=\int_0^{x_3} \mathrm{d}z \exp(2\mathrm{i}k\sigma_0 z)\left[\frac{-\mathrm{i}k\sigma_0}{2} M_\mu(z)+\frac{\mathrm{i}k}{2k\sigma_0}\left(M_\mu(z)-\frac{\langle c\rangle^2}{\sigma_0^2\beta_0^2} M_\rho(z)\right)\right]. \tag{5.8}$$

Also the formula for $G^{(2)}$ is simplified by the fact that $G^{(1)}=0$; the result is

$$G^{(2)}=\operatorname{diag}(\gamma,\gamma^*)$$

$$\gamma=\frac{-3\mathrm{i}k\sigma_0}{2}-k^2\sigma_0^2\left(\frac{\langle c\rangle^2}{\beta_0^2\sigma_0^2}\right)\lim_{z\to\infty} z^{-1}\int_0^z \mathrm{d}z\int_0^{z_1}\mathrm{d}z_2\, f(z_1,z_2)$$

$$\begin{aligned} f(z_1,z_2)=&R_{XX}^2(z_1,z_2)[1-\exp 2\mathrm{i}k\sigma_0(z_1-z_2)] \\ &+R_{YY}^2(z_1,z_2)[1-(\beta_0^4/\langle c\rangle^4)(\sigma_0^2-1)^2\exp 2\mathrm{i}k\sigma_0(z_1-z_2)]. \end{aligned} \tag{5.9}$$

Note that the wave always attenuates on the slow depth scale $\varepsilon^2 x_3$ because

$$\begin{aligned}\operatorname{Re}(\gamma)=&-k^2\sigma_0^2\left(\frac{\langle c\rangle^2}{\beta_0^2\sigma_0^2}\right)\lim_{z\to\infty} z^{-1}\int_0^z \mathrm{d}z_1\int_0^{z_1}\mathrm{d}z_2\{R_{XX}^2(z_1,z_2)[1-\cos 2k\sigma_0(z_1-z_2)] \\ &+R_{YY}^2(z_1,z_2)[1-(\beta_0^4/\langle c\rangle^4)(\sigma_0^2-1)^2\cos 2k\sigma_0(z_1-z_2)]\}<0\end{aligned}$$

since a sufficient condition for positivity of the integrand is $\langle c\rangle^4>\beta_0^4(\sigma_0^2-1)^2$ which is always satisfied because $\beta_0<\langle c\rangle$. Equations (5.5)-(5.9) yield the corrected asymptotic approximation

$$\begin{aligned} \langle V\rangle(x_3)=&V(0)\exp(\varepsilon^2 x_3\operatorname{Re}(\gamma))\{\cos[k\sigma_0-\varepsilon^2\operatorname{Im}(\gamma)]x_3 \\ &+\varepsilon\operatorname{Re}M(x_3)\cos[k\sigma_0+\varepsilon^2\operatorname{Im}(\gamma)]x_3 \\ &+\varepsilon\operatorname{Im}M(x_3)\sin[k\sigma_0+\varepsilon^2\operatorname{Im}(\gamma)]x_3\} \\ \langle T\rangle(x_3)=&-k\sigma_0\mu_0 V(0)\exp(\varepsilon^2 x_3\operatorname{Re}(\gamma)\{\sin[k\sigma_0-\varepsilon^2\operatorname{Im}(\gamma)]x_3 \\ &+\varepsilon\operatorname{Re}M(x_3)\{\sin[k\sigma_0+\varepsilon^2\operatorname{Im}(\gamma)]x_3-\varepsilon\operatorname{Im}M(x_3) \\ &-\varepsilon\operatorname{Im}M(x_3)\cos[k\sigma_0+\varepsilon^2\operatorname{Im}(\gamma)]x_3\} \end{aligned} \tag{5.10}$$

where $M(x_3)$ and γ are given by equations (5.8) and (5.9), respectively. Observe that, in contrast to the homogeneous case, the wave does not jump from pure oscillatory behaviour in $x_3\leq H$ to exponential fall-off for $x_3>H$. Instead, the transition at the boundary is essentially a change in the scale of amplitude decrease. In this sense fluctuations in the layer produce smoother physical behaviour: this remark also applies to the statistically homogeneous model since the function β in equation (4.5) is positive for any set of correlation functions.

6. The dispersion relation

So far we have not used the boundary conditions in this problem which are

(i) the displacement u_2 and stress t_{23} must be continuous at the interface $x_3=H$, and

(ii) the plane surface $x_3=0$ is free of traction, $t_{23}(0)=0$. This condition has been incorporated into the initial data of equation (2.13), $T(0)=0$. The acceptable displacement $W(x_3)$ in the half-space is just

$$W(x_3)=A\exp(-k\sigma_2 x_3) \tag{6.1}$$

for some constant A. Actually the boundary condition (i) makes A a random variable, $A=A(\omega)$ since (i) leads to

$$V(H,\omega)=A\exp(-k\sigma_2 H)$$

$$T(H,\omega)=-k\sigma_2\mu_2 A\exp(-k\sigma_2 H)$$

from which one obtains the mean value $\langle A\rangle=\exp(k\sigma_2 H)\langle V\rangle(H)$.

The dispersion relation is therefore

$$V(H,\omega)=-T(H,\omega)/k\delta_2\mu_2=-T(H,\omega)f(\langle c\rangle)$$

where

$$f(\langle c\rangle)=[k\mu_2(1-\langle c\rangle^2/\beta_2^2)^{1/2}]$$

and we use this notation since k, μ_2 and β_2 are given. Since $f(\langle c\rangle)$ is independent of ω, we may take expectations to obtain

$$f(\langle c\rangle)=-\langle V\rangle(H)/\langle T\rangle(H). \tag{6.2}$$

This equation gives the mean phase velocity $\langle c\rangle$ as a function of the wavenumber k: we shall examine the dispersion relation (6.2) for the statistically homogeneous model and compare it with the formula valid in the absence of fluctuations. Combining equations (4.4) and (6.2) yields

$$\tan(k\sigma_0+\varepsilon^2\delta)H=\mu_2\sigma_2/\mu_0\sigma_0. \tag{6.3}$$

The only difference between this and the dispersion for the homogeneous case is the factor $\varepsilon^2\delta$ on the left-hand side. Let us rewrite the last equation in terms of the dimensionless variables $x=\langle c\rangle\beta_0^{-1}$ and $k\nu$ where ν is the correlation length. For fixed $k\nu$ we have

$$\tan[f(x)+g(x)]=h(x) \qquad 1<x\leqslant\beta_2\beta_0^{-1} \tag{6.4}$$

where

$$f(x)=H'(k)(x^2-1)^{1/2}$$

$$H'=H^{-1}\nu$$

$$h(x)=(\mu_2/\mu_0)(1-x^2\beta_0^2\beta_2^{-2})^{1/2}/(x^3-1)^{1/2}$$

and we have $\varepsilon^2\ll 1$, $H'\gg 1$ with $\varepsilon^2 H'$ of order one. Notice that $f(x)$ is monotonically increasing while $h(x)$ decreases monotonically in the given range of x values. Let x_{R} denote a root of equation (6.4) and x_0 the corresponding root when $\varepsilon=0$, i.e. the scaled phase velocity of the homogeneous problem. Equation (6.4) has an infinity of roots each of which corresponds to a mode of propagation. For the nth mode we have

$$\begin{aligned} f(x_{\mathrm{R}})+g(x_{\mathrm{R}})&=\tan^{-1}(h(x_{\mathrm{R}}))+n\pi \\ f(x_0)&=\tan^{-1}(h(x_0))+n\pi. \end{aligned} \tag{6.5}$$

For a given model, i.e. a specific set of correlation functions, layer velocities, etc, one may then numerically solve equation (6.5) to obtain the dispersion curves. From the monotonic properties of f and h it follows that

$$g(x) \geqslant 0 \Rightarrow x_{\mathrm{R}} \leqslant x_0. \tag{6.6}$$

For example, if we assume that

$$R_{\mu\mu} = R(r) = R_{\rho\rho} \tag{6.7}$$

then equation (4.5) gives

$$g(x) = \tfrac{1}{2}\varepsilon^2 Hk\sigma_0[1 - 2k\sigma_0 \operatorname{Im} S(2k\sigma_0)]$$

and hence we see that the phase velocity is decreased by the fluctuations if

$$(2k\sigma_0)\int_0^\infty R(r)\sin(2k\sigma_0 r)\,\mathrm{d}r < 1. \tag{6.8}$$

One may check that this condition is satisfied for the usual exponential correlation functions. In particular

$$R(r) = \exp(-r/\nu)$$

gives

$$g(x) = \frac{\varepsilon^2 H'(k\sigma_0\nu)}{2(1 + H'k\nu\sigma_0)} > 0$$

and therefore $x_{\mathrm{R}} < x_0$. Note that the amplitude attenuation factor due to the fluctuations does not appear in the dispersion relation (6.2). In this sense the phase shift and amplitude attenuation effects are decoupled. Finally, we remark that one normally expects the inequality (6.8) to be satisfied since it is physically intuitive that the presence of random scattering terms reduces the effective propagation speed of the wave.

7. Discussion

We have studied the propagation of SH waves in a layered elastic medium whose material parameters are stochastic functions of the vertical coordinate: this is done for the case of a statistically inhomogeneous layer as well as for the simpler statistically homogeneous model considered in § 4 above. In both cases the equations of motion are solved up to a certain order in the fluctuation parameter and the effect of the random terms is broken up into an amplitude attenuation factor and a phase shift term. We find that there is no qualitative difference between the two attenuation factors. However, the dispersion relation which determines the phase velocity is very difficult to analyse in general. We focus on the dispersion for the statistically homogeneous model in comparison with the homogeneous case. In particular, we obtained a criterion which determines whether the symmetric fluctuations decrease the phase velocity.

Finally, we have restricted attention to the case of a single random layer and a homogeneous half-space. However, there is no difficulty in extending the theory to the multi-layered case: the only point worth noting is that the initial data for the second and subsequent layers is a random vector. This fact only requires the replacement of f_0 in the expansions (3.3) and (3.6) by the known quantity $\langle f_0 \rangle$.

Acknowledgments

This work was started at the Dublin Institute for Advanced Studies and completed at the University of Alberta with support from the Natural Sciences and Engineering Research Council (Canada). Comments from the referees have improved the original version.

References

Achenbach J D 1973 *Wave Propagation in Elastic Solids* (Amsterdam: North-Holland)
Aki K and Richards P G 1980 *Quantitative Seismology—Theory and Methods* (San Francisco: Freeman)
Chow P L, Kohler W and Papanicolaou G (ed) 1980 *Multiple Scattering and Waves in Random Media* (Amsterdam: North-Holland)
Frigerio A, Lewis J T and Pule J 1981 *Adv. Appl. Math.* **2** 456-81
Hudson J A 1970 *Proc. Camb. Phil. Soc.* **67** 215-23
—— 1982 *Geophys. J. R. Astron. Soc.* **69** 649-57
Ishimaru A 1982 *Wave Propagation and Scattering in Random Media* (New York: Academic)
Lancaster P 1969 *Theory of Matrices* (New York: Academic)
Lenoach B 1983a *J. Phys. A: Math. Gen.* **16** 2979-86
—— 1983b *PhD thesis* National University of Ireland
Papanicolaou G C 1978 *Studies in Probability Theory* ed M Rosenblatt (New Jersey: Mathematical Association of America)
Wentzell A 1981 *A Course in the Theory of Stochastic Processes* (New York: McGraw-Hill)